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M.A./M.Sc. FIRST SEMESTER EXAMINATION, JANUARY 2015

FIRST YEAR

MATHEMATICS

Paper : III

Date : 16/01/2015

Time : 11 am – 1 pm

Full Marks : 50

Answer any ten questions : [10×5]

1. Let $f : Y \to \mathbb{R}$ be uniformly continuous over a bounded subset Y of $\mathbb{R}$. Show that f is bounded on Y. Show that the conclusion of this result becomes false if ''boundedness of Y'' is dropped. Prove that the function $g:(0,1)\to\mathbb{R}$ given by $g(x)=\frac{1}{x}$ is not uniformly continuous over (0,1). [3+1+1]

2. For $A, B \subseteq \mathbb{R}$, define $A+B=\{a+b : a\in A, b\in B\}$.
   a) If A, B are open, show that A+B is open. [3]
   b) If A, A+B are open, does it follow that B is necessarily open? [2]

3. a) If $f_n \to f$ uniformly on $\mathbb{R}$ and $g_n \to g$ uniformly on $\mathbb{R}$, show that $\max\{f_n, g_n\} \to \max\{f, g\}$ uniformly on $\mathbb{R}$. [3]
   b) If $f_n^k \to f^k$ uniformly on $\mathbb{R}$ $\forall k \in \mathbb{N}$, does it follow that $\sup\limits_{k\in\mathbb{N}} f_n^k \to \sup\limits_{k\in\mathbb{N}} f^k$ uniformly on $\mathbb{R}$? [2]

4. Let $\sum a_n$ be a divergent series of positive real numbers. Discuss convergence / divergence of $\sum \frac{a_n}{1+n^p a_n}$ for various values of $p \geq 1$. [5]

5. State and prove Cantor's intersection theorem. Show that the conclusion of the theorem is false if we drop the assumption that the sequence of diameters of the sets converges to 0. [4+1]

6. Prove that the set of all invertible linear operators on $\mathbb{R}^n$ is open. [5]

7. a) Give an example of a sequence in $\mathbb{R}$ with countably infinite number of subsequential limits. [2]
   b) If $\{x_n\}, \{y_n\}$ are real sequences such that $x_n \leq y_n$ $\forall n \geq 100$, show that $\underline{\lim}\, x_n \leq \underline{\lim}\, y_n$. [3]

8. For $f:(a,b)\to\mathbb{R}$, define $w_f(\delta) = \sup\limits_{|x_1-x_2|\leq\delta} |f(x_1)-f(x_2)| (a \leq x_1, x_2 \leq b)$. Prove that f is uniformly continuous on (a,b) iff $\lim\limits_{\delta\to 0^+} w_f(\delta) = 0$. [5]

9. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function such that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and both are continuous. Show that f has a total derivative at each point on $\mathbb{R}^2$. [5]

10. Let $E \subseteq \mathbb{R}^{n+m}$ be open and $f:E\to\mathbb{R}^n$ be a C^1 map. Let $f(a,b)=0$ for some $(a,b)\in E$. Put $A = f'(a,b)$ and assume that A_x is invertible. Show that there exist open sets $U \subseteq \mathbb{R}^{n+m}$ and $W \subseteq \mathbb{R}^m$ with $(a,b)\in U, b\in W$ and such that $\forall y \in W, \exists !\, x\in\mathbb{R}^n$ such that $(x,y)\in U$ and $f(x,y)=0$. [5]

11. Determine as to whether the origin is a local minima / maxima or a saddle point for the following functions defined on $\mathbb{R}^2$: (i) $g_1(x,y)=xy$, (ii) $g_2(x,y)=y^2+2x^4+y^4$. [2+3]

12. a) State Weirstrass' approximation theorem. [1]
    b) Prove that the theorem follows if we can prove it in case the closed interval under consideration is [0,1] and the function vanishes at 0 and 1. [2]
    c) Show that $C^K[a,b]$ is dense in C[a,b] form any $k\in\mathbb{N}$. [2]

13. Show that the improper integral $\int_0^\infty \frac{x^{p-1}}{1+x}dx$ is convergent if and only if $0<p<1$. [5]

14. Suppose $0<\delta<\pi$, $f(x)=\begin{cases}1, & \text{if } |x|\le\delta \\ 0, & \text{if } \delta<|x|\le\pi\end{cases}$ and $f(x+2\pi)=f(x)\forall x\in\mathbb{R}$.

a) Compute the Fourier coefficients of f. [1]

b) Conclude that $\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{n}=\frac{\pi-\delta}{2}$ $(0<\delta<\pi)$. [2]

c) Deduce from Parseval's theorem that : $\sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2\delta}=\frac{\pi-\delta}{2}$ $(0<\delta<\pi)$. [2]

15. Let $f,g:[0,1]\to\mathbb{R}$ be continuous functions and $A\subseteq[0,1]$ be such that A, A^c are both dense in [0,1].

Define $h:[0,1]\to\mathbb{R}$ by $h(x)=\begin{cases}f(x) & \text{if } x\in A \\ g(x) & \text{if } x\in A^c\end{cases}$

If h is R-integrable, show that $f(x)=g(x)$ $\forall x\in[0,1]$. [5]

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